Finite Abelian Groups

G - non-trivial, finite, Abelian group

$|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, p_i distinct primes , $\alpha_i \in \mathbb{N}$

Definition $\quad G_{p_i} := \{ x \in G \mid \mathrm{Ord}(x) \text{ a power of } p_i \}$

Proposition $\quad G_{p_i} \subset G$ is a subgroup

Proof

- $\mathrm{ord}(0) = 1 = p_i^0 \Rightarrow 0 \in G_{p_i}$
- $x, y \in G_{p_i} \Rightarrow \exists\, n, m \in \mathbb{N}$ such that $p_i^n x = 0$ and $p_i^m y = 0$
  $\Rightarrow p_i^{(n+m)}(x+y) = 0 \Rightarrow \mathrm{ord}(x+y) \mid p_i^{(n+m)} \Rightarrow x+y \in G_{p_i}$
- $x \in G_{p_i} \Rightarrow \exists\, n \in \mathbb{N}$ such that $p_i^n x = 0 \Rightarrow p_i^n(-x) = 0 \Rightarrow -x \in G_{p_i}$

$\square$

Example $\quad G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z} \Rightarrow |G| = 3 \cdot 5^3$

$G_3 = \{ ([x]_3, [0]_5, [0]_{5^2}) \mid x \in \mathbb{Z} \} \cong \mathbb{Z}/3\mathbb{Z}$

$G_5 = \{ ([0]_3, [x]_5, [y]_{5^2}) \mid x \in \mathbb{Z}, y \in \mathbb{Z} \} \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z}$

Proposition $\quad |G_{p_i}| = p_i^{\alpha_i}$

Proof $\quad$ Assume $p \in \mathbb{N}$ is a prime such that $p \mid |G_{p_i}|$

Sylow's Theorem $\Rightarrow \exists\, H \subset G_{p_i}$ a subgroup such that $|H| = p$

$|H| = p \Rightarrow \exists\, x \in H$ such that $gp(x) = H \Rightarrow \mathrm{ord}(x) = p$

$\Rightarrow p = p_i \Rightarrow |G_{p_i}| = p_i^n$ for some $n \in \mathbb{N}$

Lagrange $\Rightarrow |G_{p_i}| \mid |G| \Rightarrow p_i^n \mid p_1^{\alpha_1} \dots p_n^{\alpha_n} \Rightarrow |G_{p_i}| \leq p_i^{\alpha_i}$

Sylow $\Rightarrow \exists\, H \subset G$ a subgroup such that $|H| = p_i^{\alpha_i}$

$x \in H \Rightarrow \operatorname{ord}(x) \,\big|\, |H| \Rightarrow \operatorname{ord}(x) \,\big|\, p_i^{\alpha_i} \Rightarrow x \in G_{p_i} \Rightarrow H \subset G_{p_i}$

$\Rightarrow |H| \leq |G_{p_i}| \Rightarrow p_i^{\alpha_i} \leq |G_{p_i}|$

$\Rightarrow |G_{p_i}| = p_i^{\alpha_i}$

$\square$

<u>Theorem</u> $G \cong G_{p_1} \times G_{p_2} \times \dots \times G_{p_n}$

<u>Proof</u> Let $\phi : G_{p_1} \times G_{p_2} \times \dots \times G_{p_n} \longrightarrow G$

$(x_1, x_2, \dots x_n) \longmapsto x_1 + x_2 + \dots + x_n$

composition in G

G Abelian $\Rightarrow$ ϕ a homomorphism

$\operatorname{Ker}\phi = \{ (x_1, \dots, x_n) \mid x_1 + \dots + x_n = e \}$

just notation

$x_1 x_2 \dots x_n = e \Rightarrow -x_1 = x_2 + x_3 + \dots + x_n = y$

$x_i \in G_{p_i} \Rightarrow p_i^{\alpha_i} x_i = 0$

$p_1^{\alpha_1} y = p_1^{\alpha_1}(-x_1) = -(p_1^{\alpha_1} x_1) = 0$

$$\begin{aligned} p_2^{\alpha_2} \dots p_n^{\alpha_n} y &= p_2 \dots p_n^{\alpha_n} (x_2 + \dots + x_n) \\ &= \underline{p_2}^{\alpha_2} \dots p_n^{\alpha_n} \underline{x_2} + p_2^{\alpha_2} \underline{p_3}^{\alpha_3} \dots p_n^{\alpha_n} \underline{x_3} + \dots + p_2^{\alpha_2} \dots \underline{p_n}^{\alpha_n} \underline{x_n} \\ &= 0 + 0 + \dots + 0 = 0 \end{aligned}$$

$\Rightarrow \operatorname{Ord}(y) \mid p_1^{\alpha_1}$ and $\operatorname{Ord}(y) \mid p_2^{\alpha_2} \dots p_n^{\alpha_n} \Rightarrow \operatorname{Ord}(y) = 1 \Rightarrow y = 0$

$\Rightarrow x_1 = 0$

We can apply this logic to each x_i.

$\Rightarrow \operatorname{Ker}\phi = \{(0, 0, \dots, 0)\} \Rightarrow \phi$ injective

$|G_{p_1} \times G_{p_2} \times \dots \times G_{p_n}| = |G_{p_1}| \times |G_{p_2}| \times \dots \times |G_{p_n}| = p_1^{\alpha_1} \dots p_n^{\alpha_n} = |G|$

$\Rightarrow \phi$ bijective $\Rightarrow \phi$ an isomorphism

$\square$

Theorem Let G be a finite Abelian group and $p \in \mathbb{N}$ be a prime such that $G = G_p$. every element of G has order a power of p

$$G \cong \mathbb{Z}/p^{n_1}\mathbb{Z} \times \mathbb{Z}/p^{n_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_k}\mathbb{Z}$$

Moreover the numbers $\{n_1, n_2, \ldots, n_k\}$ are unique up to reordering.

$|H_i| = p^{n_i} \Rightarrow H \cong \mathbb{Z}/p^{n_i}\mathbb{Z}$

Proof (Vague Outline)

Strategy : Find $H_1, \ldots, H_k \subset G$ cyclic subgroups such that

$$G = H_1 \oplus \cdots \oplus H_k \cong \mathbb{Z}/p^{n_1}\mathbb{Z} \times \mathbb{Z}/p^{n_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_k}\mathbb{Z}$$

See notes for details $\square$

Structure Theorem for Finitely Generated Abelian Groups

Let G be a finitely generated Abelian group. Then $\exists\, C_1, \ldots, C_n$ cyclic groups such that

1/ $G \cong C_1 \times C_2 \cdots \times C_n$

2/ C_i is either $(\mathbb{Z}, +)$ or $(\mathbb{Z}/p^n\mathbb{Z}, +)$ for some prime p.

3/ If $G \cong D_1 \times \cdots \times D_m$ is another such expansion then $\underline{n = m}$ and (perhaps after reordering) $C_i = D_i$

Proof (Outline)

finite

G f.g. Abelian $\Rightarrow G \cong \mathbb{Z}^n \times tG$

$|tG| = p_1^{\alpha_1} \cdots p_n^{\alpha_n} \Rightarrow tG \cong (tG)_{p_1} \times \cdots \times (tG)_{p_n}$ depends on i

$(tG)_{p_i} \cong \mathbb{Z}/p_i^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_i^{n_k}\mathbb{Z}$ for some $k \in \mathbb{N}$

$\square$

Example $G \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

$\Rightarrow$ rank$(G) = 2$, $tG \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

$(tG)_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z}$, $(tG)_3 \cong \mathbb{Z}/3\mathbb{Z}$, $(tG)_5 \cong \mathbb{Z}/5\mathbb{Z}$